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Poiseuille and Thermal-Creep Flow in a Cylindrical Tube

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A version of the discrete-ordinates method is used to solve, for the case of flow in a cylindrical tube, the classical Poiseuille and thermal-creep problems based on the Bhatnagar, Gross, and Krook model in the theory of rarefied-gas dynamics. In addition to the development of a discrete-ordinates solution that is valid for a wide range of the Knudsen number, the solution is evaluated numerically for selected cases to yield results, thought to be correct to many significant figures, for the slip velocities, the macroscopic velocity profiles, and the flow rates. © 2000 Academic Press

1. INTRODUCTION

In a recent work [1] a newly developed version [2] of the discrete-ordinates method [3] was used to solve in a definitive manner the most important of the classical, plane-geometry flow problems defined in terms of the Bhatnagar *et al.* model [4] basic to the general area of rarefied-gas dynamics [5, 6]. In this work, we extend that earlier work [1] to the cases of Poiseuille flow and thermal creep in a cylindrical tube. Since much of what we use in this work was developed in Refs. [1] and [2] and in a paper by Valougeorgis and Thomas [7], our presentation here is brief.

We start with a mathematical formulation of the problems we intend to solve in this work. This formulation was developed by Ferziger [8] and Loyalka [9] and was used by Valougeorgis and Thomas [7] as a starting point in developing a solution, based on the F_N method [10], to the same problems we consider here. And so we begin with the integral equation

$$Z(r) = \int_0^R tZ(t)K(t \to r)\,\mathrm{d}t + S(r) \tag{1}$$

for $r \in [0, R]$. Here, as will be explicitly noted, the basic unknown $Z(r)$ is related to the

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desired macroscopic velocity profile, $S(r)$ is a specified inhomogeneous source term, and

$$K(t \to r) = 2\pi^{-1/2} \int_0^\infty \mathrm{e}^{-u^2} K_0(r/u) I_0(t/u) \frac{\mathrm{d}u}{u^2}, \quad t \in [0, r], \tag{2a}$$

and

$$K(t \to r) = 2\pi^{-1/2} \int_0^\infty \mathrm{e}^{-u^2} K_0(t/u) I_0(r/u) \frac{\mathrm{d}u}{u^2}, \quad t \in [r, R], \tag{2b}$$

where we use $I_n(x)$ and $K_n(x)$ to denote the modified Bessel functions of the first and second kind [11]. Following previously mentioned works [7–9], we note that for the problem of Poiseuille flow the source term in Eq. (1) is

$$S_P(r) = \frac{1}{2}\pi^{1/2}. \tag{3}$$

In regard to the quantities of physical interest that we wish to establish and in order to be consistent with previous works to which we wish to compare our numerical results, we follow the definitions from Refs. [7] and [12] and thus will compute the macroscopic velocity profile

$$q_P(r) = \pi^{-1/2} Z_P(r) - \frac{1}{2} \tag{4}$$

and the flow rate

$$Q_P = \frac{4}{R^3} \int_0^R q_P(r) r \, \mathrm{d}r. \tag{5}$$

On the other hand, for the thermal-creep problem the source term in Eq. (1) is given [7] by

$$S_T(r) = R \int_0^\infty u K_1(R/u) I_0(r/u) \mathrm{e}^{-u^2} \, \mathrm{d}u, \tag{6}$$

and again, following previously defined [7, 12] quantities of physical interest, we intend to compute the macroscopic velocity profile

$$q_T(r) = \pi^{-1/2} Z_T(r) - \frac{1}{4} \tag{7}$$

and the flow rate

$$Q_T = \frac{4}{R^3} \int_0^R q_T(r) r \, \mathrm{d}r. \tag{8}$$

Note that we have added the subscripts P and T to distinguish between the Poiseuille-flow and the thermal-creep problems. We note also that Ref. [12] provides numerical results for the considered problems with an arbitrary mixture of specular and diffuse reflection of particles from the wall of the tube, and so we should make it clear that this work here is restricted to the case of purely diffuse reflection [6].

2. A REFORMULATION OF THE PROBLEM

As was done some years ago in two works concerning neutron-transport theory in cylindrical geometry [13, 14], we make use of a convenient transformation [15] to reformulate the problems defined by Eq. (1) in terms of a "pseudo problem" for which we can use much of our experience with flow problems for plane channels. And so, as did Valougeorgis and Thomas [7], we let

$$\Phi(r,\xi) = \xi^{-2}\left[K_0(r/\xi)\int_0^r tZ(t)I_0(t/\xi)\,\mathrm{d}t + I_0(r/\xi)\int_r^R tZ(t)K_0(t/\xi)\,\mathrm{d}t\right] \tag{9}$$

which we can differentiate twice and use along with Eq. (1) to find that $\Phi(r,\xi)$ satisfies

$$\xi^2\frac{\partial^2}{\partial r^2}\Phi(r,\xi) + \frac{\xi^2}{r}\frac{\partial}{\partial r}\Phi(r,\xi) - \Phi(r,\xi) + 2\int_0^\infty \Psi(u)\Phi(r,u)\,\mathrm{d}u = -S(r), \tag{10}$$

for $\xi \in (0,\infty)$, and that

$$Z(r) = 2\int_0^\infty \Psi(\xi)\Phi(r,\xi)\,\mathrm{d}\xi + S(r). \tag{11}$$

Here we use

$$\Psi(u) = \pi^{-1/2}\mathrm{e}^{-u^2}. \tag{12}$$

The definition of $\Phi(r,\xi)$ as given by Eq. (9) allows us to deduce a boundary condition subject to which we intend to solve Eq. (10), viz.,

$$\Phi(R,\xi) + \xi\Gamma(\xi)\frac{\partial}{\partial r}\Phi(r,\xi)\bigg|_{r=R} = 0 \tag{13}$$

for $\xi \in (0,\infty)$. Here

$$\Gamma(\xi) = \frac{K_0(R/\xi)}{K_1(R/\xi)}. \tag{14}$$

At this point we can use the particular solutions reported by Valougeorgis and Thomas [7], viz.

$$G_P(r,\xi) = -\frac{1}{4}\pi^{1/2}(r^2 - R^2 + 4\xi^2) \tag{15a}$$

for the Poiseuille-flow problem and

$$G_T(r,\xi) = -\frac{1}{2}\pi^{1/2}R\xi K_1(R/\xi)I_0(r/\xi) \tag{15b}$$

for the thermal-creep problem to obtain a homogeneous version of Eq. (10). And so we substitute the general decomposition

$$\Phi(r,\xi) = Y(r,\xi) + G(r,\xi) \tag{16}$$

into Eqs. (10) and (13) to obtain

$$\xi^2 \frac{\partial^2}{\partial r^2} Y(r,\xi) + \frac{\xi^2}{r}\frac{\partial}{\partial r} Y(r,\xi) - Y(r,\xi) + 2\int_0^\infty \Psi(u) Y(r,u)\,\mathrm{d}u = 0 \tag{17}$$

for $\xi \in (0,\infty)$ and the boundary condition

$$Y(R,\xi) + \xi\Gamma(\xi)\frac{\partial}{\partial r} Y(r,\xi)\bigg|_{r=R} = F(\xi) \tag{18}$$

for $\xi \in (0,\infty)$. Here

$$F_P(\xi) = \frac{1}{2}\pi^{1/2}\xi[2\xi + R\Gamma(\xi)] \tag{19a}$$

for the Poiseuille-flow problem and

$$F_T(\xi) = \frac{1}{2}\pi^{1/2}\xi^2 \tag{19b}$$

for the thermal-creep problem.

At this point we can use Eqs. (11), (12), (15a), and (16) in Eq. (4) to express the macroscopic velocity profile for the Poiseuille-flow problem in terms of $Y(r,\xi)$, viz.

$$q_P(r) = Y_P(r) - \frac{1}{4}(r^2 - R^2 + 2), \tag{20}$$

and so using Eq. (5), we express the flow rate as

$$Q_P = \frac{4}{R^3}\int_0^R Y_P(r) r\,\mathrm{d}r + \frac{1}{4R}(R^2 - 4). \tag{21}$$

Here

$$Y_P(r) = 2\pi^{-1/2}\int_0^\infty \Psi(\xi) Y_P(r,\xi)\,\mathrm{d}\xi. \tag{22}$$

In a similar way we can use Eqs. (11), (12), (15b), and (16) in Eq. (7) to obtain

$$q_T(r) = Y_T(r) - \frac{1}{4} \tag{23}$$

and

$$Q_T = \frac{4}{R^3}\int_0^R Y_T(r) r\,\mathrm{d}r - \frac{1}{2R}, \tag{24}$$

where

$$Y_T(r) = 2\pi^{-1/2}\int_0^\infty \Psi(\xi) Y_T(r,\xi)\,\mathrm{d}\xi. \tag{25}$$

Having expressed the quantities we wish to compute in terms of $Y(r,\xi)$ we proceed now to develop our discrete-ordinates solution to the problem defined by Eqs. (17) and (18).

3. A DISCRETE-ORDINATES SOLUTION

To start we approximate the integral term in Eq. (17) by a quadrature formula and write our discrete-ordinates equations as

$$\xi_i^2 \frac{d^2}{dr^2} Y(r, \xi_i) + \frac{\xi_i^2}{r} \frac{d}{dr} Y(r, \xi_i) - Y(r, \xi_i) + 2 \sum_{k=1}^{N} w_k \Psi(\xi_k) Y(r, \xi_k) = 0 \tag{26}$$

for $i = 1, 2, \ldots, N$. In writing Eq. (26) as we have, we are considering that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \infty)$. Seeking a Bessel function solution (bounded as $r \to 0$) of Eq. (26), we substitute

$$Y(r, \xi_i) = \phi(\nu, \xi_i) I_0(r/\nu) \tag{27}$$

into Eq. (26) to find

$$\left(1 - \frac{\xi_i^2}{\nu^2}\right) \phi(\nu, \xi_i) = 2 \sum_{k=1}^{N} w_k \Psi(\xi_k) \phi(\nu, \xi_k) \tag{28}$$

for $i = 1, 2, \ldots, N$. Now if we let $\phi(\nu, \xi_k)$, $k = 1, 2, \ldots, N$, define the elements of an N vector $\boldsymbol{\Phi}(\nu)$ we can rewrite Eq. (28) as

$$(\boldsymbol{I} - \lambda \boldsymbol{M}^2) \boldsymbol{\Phi}(\nu) = 2 \boldsymbol{W} \boldsymbol{\Phi}(\nu), \tag{29}$$

where $\lambda = 1/\nu^2$, $\boldsymbol{I}$ is the $N \times N$ identity matrix, the elements of $\boldsymbol{W}$ are given by

$$(\boldsymbol{W})_{i,j} = w_j \Psi(\xi_j), \tag{30}$$

and

$$\boldsymbol{M} = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}. \tag{31}$$

We note that, not surprisingly, the eigenvalue problem defined by Eq. (29) is the same as the one encountered in Refs. [1] and [2] in the discrete-ordinates solutions of equivalent problems in plane geometry, and so we take advantage of those works and rewrite Eq. (29) in the special [16] form

$$(\boldsymbol{D} - 2\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}}) \boldsymbol{X} = \lambda \boldsymbol{X}, \tag{32}$$

where, again, $\lambda = 1/\nu^2$,

$$\boldsymbol{D} = \mathrm{diag}\left\{\xi_1^{-2}, \xi_2^{-2}, \ldots, \xi_N^{-2}\right\}, \tag{33}$$

and

$$\boldsymbol{z} = \left[\frac{\sqrt{w_1 \Psi(\xi_1)}}{\xi_1}, \frac{\sqrt{w_2 \Psi(\xi_2)}}{\xi_2}, \ldots, \frac{\sqrt{w_N \Psi(\xi_N)}}{\xi_N}\right]^{\mathrm{T}}. \tag{34}$$

Here we use the superscript T to denote the transpose operation. Continuing, we note that the eigenvalue problem defined by Eq. (32) is of a form that is encountered when the so-called "divide and conquer" method [17] is used to find the eigenvalues of tridiagonal matrices. In addition, we see from Eq. (33) that, because of the way our basic eigenvalue problem is formulated, we must exclude zero from the set of quadrature points. Of course to exclude zero from the quadrature set is not considered a serious restriction since typical Gauss quadrature schemes do not include the end points of the integration interval.

Now, considering that we have found the eigenvalues defined by Eq. (32) and the required separation constants from

$$\nu_j = \lambda_j^{-1/2} \tag{35}$$

for $j = 1, 2, \dots, N$, we impose the normalization condition

$$2\sum_{k=1}^{N} w_k \Psi(\xi_k)\phi(\nu, \xi_k) = 1 \tag{36}$$

so we can write

$$\phi(\nu_j, \xi_i) = \frac{\nu_j^2}{\nu_j^2 - \xi_i^2}, \tag{37}$$

where clearly, as discussed later in Section 5, we cannot allow $\nu_j = \xi_i$. Continuing, we "sum up" our solutions and write

$$Y(r, \xi_i) = \sum_{j=1}^{N} A_j \phi(\nu_j, \xi_i) I_0(r/\nu_j), \tag{38}$$

where the arbitrary constants $\{A_j\}$ are to be determined from the boundary condition of our problem.

At this point we wish to modify slightly the discrete-ordinates solution given by Eq. (38). We note that problems based on Eq. (17) are "conservative" since

$$2\int_0^\infty \Psi(\xi)\,\mathrm{d}\xi = 1, \tag{39}$$

and so we expect that one of the eigenvalues defined by Eq. (32) should tend to zero as N tends to infinity. We choose to take this fact into account by explicitly neglecting ν_N, the largest of the computed separation constants $\{\nu_j\}$ and, subsequently, by writing Eq. (38) as

$$Y(r, \xi_i) = A + \sum_{j=1}^{N-1} A_j \phi(\nu_j, \xi_i) \hat{I}_0(r/\nu_j) \mathrm{e}^{-(R-r)/\nu_j}. \tag{40}$$

Of course, the constants A and $\{A_j\}$ in Eq. (40) are to be determined by constraining $Y(r, \xi_i)$ to meet a discrete-ordinates version of the relevant boundary condition. To complete our discussion of Eq. (40) we note that we have "rescaled" the solution by introducing (in general)

$$\hat{I}_n(x) = I_n(x)\mathrm{e}^{-x} \tag{41a}$$

and (to be used later)

$$\hat{K}_n(x) = K_n(x)\mathrm{e}^x \tag{41b}$$

in order to keep “underflows/overflows” in our numerical work from degrading our calculation.

To conclude this section, we note first of all that we can use the discrete-ordinates solution given by Eq. (40) in the definition

$$Y(r) = 2\pi^{-1/2}\int_0^\infty \Psi(\xi)Y(r,\xi)\,\mathrm{d}\xi \tag{42}$$

to obtain, after we note Eq. (36), the discrete-ordinates results

$$Y(r) = \pi^{-1/2}\left[A + \sum_{j=1}^{N-1} A_j \hat{I}_0(r/\nu_j)\mathrm{e}^{-(R-r)/\nu_j}\right] \tag{43}$$

and

$$\frac{4}{R^3}\int_0^R Y(r)r\,\mathrm{d}r = \frac{2\pi^{-1/2}}{R^2}\left[AR + 2\sum_{j=1}^{N-1} A_j\nu_j\hat{I}_1(R/\nu_j)\right]. \tag{44}$$

Having developed our discrete-ordinates formalism, we are ready to solve the Poiseuille and thermal-creep problems concerning flow in a cylindrical tube.

4. SOLUTIONS TO THE PROBLEMS

To complete the solutions to the considered problems we now must determine the arbitrary constants A and $\{A_j\}$ in the general expression given by Eq. (40). And so we substitute Eq. (40) into Eq. (18) evaluated at the quadrature points to obtain

$$A + \sum_{j=1}^{N-1} M_{i,j}A_j = F(\xi_i) \tag{45}$$

for $i = 1, 2, \ldots, N$. Here

$$M_{i,j} = \nu_j\left[\frac{\nu_j\hat{I}_0(R/\nu_j) + \xi_i\Gamma(\xi_i)\hat{I}_1(R/\nu_j)}{\nu_j^2 - \xi_i^2}\right] \tag{46}$$

and $F(\xi_i)$ is either

$$F_P(\xi_i) = \frac{1}{2}\pi^{1/2}\xi_i[2\xi_i + R\Gamma(\xi_i)] \tag{47a}$$

for the Poiseuille-flow problem or

$$F_T(\xi_i) = \frac{1}{2}\pi^{1/2}\xi_i^2 \tag{47b}$$

for the thermal-creep problem. In addition (for computational reasons) we use Eq. (41b) to write $\Gamma(\xi_i)$ as

$$\Gamma(\xi_i) = \frac{\hat{K}_0(R/\xi_i)}{\hat{K}_1(R/\xi_i)}. \tag{48}$$

Now, of course, all we have to do is to define a quadrature scheme, solve the eigenvalue problem defined by Eq. (32), thus obtaining the separation constants $\{\nu_j\}$, and solve the linear system defined by Eq. (45). In this way all that we seek here is established, viz.

$$q_P(r) = \pi^{-1/2}\left[A + \sum_{j=1}^{N-1} A_j \hat{I}_0(r/\nu_j)\mathrm{e}^{-(R-r)/\nu_j}\right] - \frac{1}{4}(r^2 - R^2 + 2) \tag{49}$$

and

$$Q_P = \frac{2\pi^{-1/2}}{R^2}\left[AR + 2\sum_{j=1}^{N-1} A_j \nu_j \hat{I}_1(R/\nu_j)\right] + \frac{1}{4R}(R^2 - 4) \tag{50}$$

for Poiseuille flow and

$$q_T(r) = \pi^{-1/2}\left[A + \sum_{j=1}^{N-1} A_j \hat{I}_0(r/\nu_j)\mathrm{e}^{-(R-r)/\nu_j}\right] - \frac{1}{4} \tag{51}$$

and

$$Q_T = \frac{2\pi^{-1/2}}{R^2}\left[AR + 2\sum_{j=1}^{N-1} A_j \nu_j \hat{I}_1(R/\nu_j)\right] - \frac{1}{2R} \tag{52}$$

for thermal-creep flow. In order to be very clear, we note that the constants A and $\{A_j\}$ in Eqs. (49) and (50) correspond to the solutions of the linear system defined by using $F_P(\xi_i)$ for $F(\xi_i)$ in Eq. (45), and likewise the constants A and $\{A_j\}$ in Eqs. (51) and (52) correspond to the solutions of the linear system defined by using $F_T(\xi_i)$ in Eq. (45).

5. NUMERICAL RESULTS

Repeating the discussion given in Ref. [1], we note that what we must now do is to define the quadrature scheme to be used in our discrete-ordinates solution. In this work we have used one of the (nonlinear) transformations

$$u(\xi) = \exp\{-\xi\} \tag{53a}$$

or

$$u(\xi) = \frac{1}{1+\xi} \tag{53b}$$

to map $\xi \in [0, \infty)$ into $u \in [0, 1]$, and we then used a Gauss–Legendre scheme mapped (linearly) onto the interval [0, 1]. Of course other quadrature schemes could be used. In fact we note that recent works by Garcia [18] and Gander and Karp [19] have reported special quadrature schemes for use in the general area of particle transport theory. Such an approach clearly could be used here. In fact the choice of a quadrature scheme based on the integration interval $[0, \infty)$ with a weight function as defined by Eq. (12) seems a natural choice for this work. However, we have found the use of a mapping defined by either of Eqs. (53) followed by the use of the Gauss–Legendre integration formulas to be so effective that we have not developed any special-purpose quadrature schemes.

Continuing the discussion from Ref. [1], we note that having defined our quadrature scheme and in developing a FORTRAN implementation of our solution, we found the required separation constants $\{\nu_j\}$ by using the special numerical package DZPACK [16] that was developed to take advantage of the special structure of Eq. (32) to solve the eigenvalue problem defined by Eq. (32). The required separation constants were then available as the reciprocals of the square roots of these eigenvalues. We then used the subroutines DGECO and DGESL from the LINPACK package [20] to solve the linear system defined by Eq. (45), and so the solutions to the various problems were considered established.

Finally, but importantly, we note that since the function $\Psi(u)$ defined by Eq. (12) can be zero, from a computational point-of-view, we can have some, say a total of N_0, of the separation constants $\{\nu_j\}$ equal to some of the quadrature points $\{\xi_i\}$. Of course this is not allowed in Eq. (37), and so, since the quadrature points where $\Psi(\xi_k)$ is effectively zero make no contribution to the right-hand side of Eq. (28), we have seen that we can simply omit these quadrature points from our calculation. Of course, in omitting these N_0 quadrature points we have effectively changed N to $N - N_0$ in some parts of our solution.

In order to illustrate the achieved accuracy of our developed discrete-ordinates solutions to the considered problems we list some typical results in Tables I and II. We note that these numerical results are given with what we believe to be seven figures of accuracy.

TABLE I
The Velocity Profiles $q_P(r)$ and $q_T(r)$ for $R = 2$

r/R	$q_P(r)$	$q_T(r)$
0.00	2.353331	2.970292(−1)
0.05	2.350206	2.967964(−1)
0.10	2.340825	2.960952(−1)
0.15	2.325161	2.949165(−1)
0.20	2.303169	2.932454(−1)
0.25	2.274788	2.910597(−1)
0.30	2.239928	2.883297(−1)
0.35	2.198478	2.850168(−1)
0.40	2.150292	2.810714(−1)
0.45	2.095184	2.764308(−1)
0.50	2.032917	2.710155(−1)
0.55	1.963187	2.647240(−1)
0.60	1.885600	2.574255(−1)
0.65	1.799630	2.489483(−1)
0.70	1.704562	2.390608(−1)
0.75	1.599385	2.274398(−1)
0.80	1.482595	2.136108(−1)
0.85	1.351773	1.968245(−1)
0.90	1.202532	1.757494(−1)
0.95	1.024896	1.474303(−1)
1.00	7.651726(−1)	9.662684(−2)

TABLE II
The Microscopic Velocity Slips $q_P(R)$ and $q_T(R)$ and the Flow Rates Q_P and Q_T

R	$q_P(R)$	$q_T(R)$	Q_P	Q_T
1.0(−2)	5.482193(−3)	2.646031(−3)	1.476313	7.178339(−1)
2.0(−2)	1.076826(−2)	5.068533(−3)	1.460303	6.959018(−1)
3.0(−2)	1.591934(−2)	7.331779(−3)	1.448271	6.781576(−1)
4.0(−2)	2.096360(−2)	9.466208(−3)	1.438589	6.629314(−1)
5.0(−2)	2.591873(−2)	1.149131(−2)	1.430520	6.494603(−1)
7.0(−2)	3.560833(−2)	1.526599(−2)	1.417717	6.262043(−1)
9.0(−2)	4.505667(−2)	1.873396(−2)	1.407986	6.064104(−1)
1.0(−1)	4.970472(−2)	2.036966(−2)	1.403962	5.974787(−1)
3.0(−1)	1.362138(−1)	4.443342(−2)	1.376211	4.824054(−1)
5.0(−1)	2.162008(−1)	5.939122(−2)	1.386652	4.170682(−1)
7.0(−1)	2.929854(−1)	6.971188(−2)	1.410539	3.713404(−1)
9.0(−1)	3.678629(−1)	7.722336(−2)	1.441274	3.364313(−1)
1.0	4.048069(−1)	8.024270(−2)	1.458291	3.217264(−1)
1.5	5.864481(−1)	9.069215(−2)	1.553226	2.655915(−1)
2.0	7.651726(−1)	9.662684(−2)	1.657647	2.271179(−1)
3.0	1.119114	1.026102(−1)	1.879988	1.766334(−1)
3.5	1.295371	1.041933(−1)	1.994994	1.589970(−1)
4.0	1.471454	1.052977(−1)	2.111623	1.445407(−1)
5.0	1.823461	1.066763(−1)	2.348327	1.222287(−1)
6.0	2.175514	1.074573(−1)	2.588211	1.058073(−1)
7.0	2.527721	1.079357(−1)	2.830249	9.322240(−2)
9.0	3.232634	1.084620(−1)	3.318540	7.522751(−2)
1.0(1)	3.585304	1.086153(−1)	3.564118	6.858111(−2)
1.0(2)	3.539559(1)	1.092709(−1)	2.602162(1)	7.582959(−3)

Of course, we have no proof of the accuracy of our results, but we have done various things to establish the confidence we have. For example, we have increased the value of N used in our computations until we found stability in the final results. We have also used numerical linear-algebra packages other than those mentioned and both nonlinear maps given by Eqs. (53) to obtain the same results as given in our tables. While we have found agreement (that varied from three to five significant figures) with relevant results from Refs. [7] and [12], we believe the results reported here should be considered more definitive than the already mentioned earlier results [7, 12].

We note that we have typically used $N = 100$ to generate the results listed in our tables, and to have an idea of the computational time required to solve both the Poiseuille-flow and the thermal-creep problems for a typical case, we note that our FORTRAN implementation (no special effort was made to make the code especially efficient) of our discrete-ordinates solutions (with $N = 100$) runs in less than a second on a 166 MHz Pentium-based notebook PC. Finally, to have some idea about N_0, the number of quadrature points not included in some parts of our calculation, we note that using $\epsilon = 10^{-14}$ to decide if an eigenvalue and a quadrature point were the same “computationally,” we found $N_0 = 3$ when $N = 100$ and the map defined by Eq. (53a) were used.

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